Journal of Statistical Physics, Vol. 104, Nos. 5/6, September 2001 (© 2001)

The Weierstrass–Mandelbrot Process Revisited[1]

Jerzy Szulga[2] and Fred Molz[3]

Received July 14, 1997; revised March 30, 2001

We derive a functional central limit theorem for quasi-Gaussian processes. In particular, we prove that the limit of the Mandelbrot–Weierstrass process is a complex fractional Brownian motion.

KEY WORDS: Fractional Brownian motion; Brownian Motion; Weierstrass–Mandelbrot process; stationary increments; structure functions; Gaussian processes; self-similar processes; quasi-Gaussian processes; spectral representation; power spectrum.

1. INTRODUCTION

1.1. Fractional Motions

A *fractional Brownian motion* (fBm, in short) $X(t)$ is a self-similar Gaussian process with stationary increments in the complex plane, or even in a higher dimensional space. For some exponent $H \in [0, 1]$, the time-scaled process $X(\lambda t)$ and the range-scaled process $|\lambda|^H X(t)$ are statistically indistinguishable, for every factor λ. In the real case, $X(t)$ is a classical Brownian motion when $H = 1/2$. The concept was introduced by Kolmogorov [Kol40] and discussed further by [MVN68] and many others.

Let ϕ_n be independent random variables uniformly distributed on $[0, 2\pi]$. The Mandelbrot–Weierstrass process (WM-process, in short)

$$M_r(t) = \sum_{n=-\infty}^{\infty} (1 - e^{ir^n t})\, r^{-Hn}\, e^{i\phi_n}, \qquad r > 1, \tag{1.1}$$

[1] The research was supported by NSF Grant 9706048.

[2] Department of Mathematics, Auburn University, Alabama.

[3] Department of Environmental Sciences, Clemson University, South Carolina.

0022-4715/01/0900-1317$19.50/0

has also stationary increments but its self-similarity is restricted to factors from the Weierstrass spectrum, $\lambda = r^m$, where m are integers. In the shape of the series one may recognize the archetype of continuity without differentiability. The original Weierstrass function has utilized the series restricted to $n \geqslant 0$ and involved deterministic phases $\phi_n = n\pi\phi$, for an irrational ϕ. Mandelbrot [Man77] extended the spectrum to low frequencies and noted a number of similarities between the randomized process and the fBm.

Berry and Lewis [BL80] observed that the variances (called here structure functions; see the vicinity of (2.9)) of the suitably normalized WM-process

$$W_r(t) = \sqrt{\ln r}\, M_r(t) \tag{1.2}$$

and of a fractional Brownian motion $W(t)$ are approximately equal, as $r \to 1$. More precisely,

$$\begin{aligned} \langle |W_r(\tau+t) - W_r(\tau)|^2 \rangle &= 2 \ln r \sum_{n=-\infty}^{\infty} (1 - \cos(r^n t))\, r^{-2Hn} \\ &\approx c_H\, |t|^{2H} = c_H \langle |W(\tau+t) - W(\tau)|^2 \rangle. \end{aligned} \tag{1.3}$$

This fact indicated that a normalized WM-process should converge to a fBm, for $r \to 1$.

Many authors have applied the WM-process to approximate the fractional Brownian motion (e.g., see [HB89, Chu93, MW69a] and references in there). This model spread to many areas, from hydrology to geophysics to astronomy to finances, and so on. The listing [Kle74, LK77, Tur89, MW68, MW69b, MW69c, MVN68, Vos89, HB89, MB93, Lo91, Kor92, MSL97] shows but a small sample of articles.

1.2. Objective

We will render Berry and Lewis's claim rigorous.

Regarding the limit as $r \to 1$, finite dimensional distributions of a process $W_r(t) = X_r(t) + iY_r(t)$ are said to *converge* to finite dimensional distributions of a process $W(t) = X(t) + iY(t)$ if, for every n, every times $t_1, \ldots, t_n$, and every real numbers $x_1, \ldots x_n, y_1, \ldots, y_n$,

$$\begin{aligned} &\lim_{r \to 1} \mathsf{P}(X_r(t_k) < x_k, Y_r(t_k) < y_k, k = 1, \ldots, n) \\ &\qquad = \mathsf{P}(X(t_k) < x_k, Y(t_k) < y_k, k = 1, \ldots, n). \end{aligned} \tag{1.4}$$

From now on, the phrases 'a process converges' or 'a limit process' refer to this definition.

The qualitative character of the definition gives no irrefutable answer to the question "How close are the processes to each other?" The simple deviation between structure functions quantitates the proximity of processes quite well but it does not relate to the proximity of distributions, in general.

Yet, for real Gaussian processes, both notions of convergence are equivalent. The equivalence usually fails for complex processes, even Gaussian processes, or if one of them is not Gaussian. The Weierstrass–Mandelbrot process was declared as a Gaussian process in [BL80] yet it is not. No Gaussian variable can be written as the sum of independent random variables of which one is not Gaussian (by Cramér's theorem, cf., e.g., [Lu70, Theorem 8.2.1]). Of course, had $W_r(t)$ been a real Gaussian process, then the convergence of its structure function to the structure function of the fBm(H) would be sufficient, and $\Re W_r(t)$ would converge to the a real fBm in distribution. For example, $W_r(t)$ shares its variance with the Gaussian process

$$G_r(t) = \sqrt{\ln r} \sum_{n=-\infty}^{\infty} (1 - e^{ir^n t})\, r^{-Hn} \frac{\xi_n + i\xi'_n}{\sqrt{2}}, \tag{1.5}$$

where $(\xi_n, \xi'_n; n \geqslant 0)$ are independent pairs of independent standard Gaussian random variables. Limit property (1.3) implies that $\Re G_r(t)$ and $\Im G_r(t)$ converge to real fBms, for $r \to 1$. But a process is not Gaussian until proven Gaussian. In particular, the sum of independent infinitesimal factors leads often but not always to a Gaussian law. For example, the Rademacher series $S = \sum_{n=1}^{\infty} R_n r^{-n}$, where Rademacher variables R_n are independent random signs ± 1, may have a variety of distributions, some of them being very far from Gaussian, for some ratios r. For example, if $r = 2$, then S is uniformly distributed on $[-1, 1]$. If $r = 3$, then the probability measure induced by S is singular because the distribution function of S has the ternary Cantor set on $[0, 1]$ as its spectrum, etc. (see [Kaw72, 13.4 & references]).

Mandelbrot [Man77, pp. 328-331] called (1.5) the Gauss–Weierstrass process, while its counterpart, with ξ_n, ξ'_n being Rademacher functions, the Weierstrass–Rademacher process (consequently, the WM-process should be called Weierstrass–Steinhaus). Nevertheless, the specific structure of the Weierstrass–Mandelbrot (or its counterparts) process will lead to the Gaussian limit.

It is well established that the Hausdorff (fractal) dimension of a typical path of real fBm is equal to $D = 2 - H$, and this was mentioned already by

Mandelbrot. In [BL80] the authors gave an heuristic and experimental evidence that the real part of the WM-process should also have trajectories with $D_{Hausdorff} = 2 - H$. The key step in establishing the result was to prove that the electrostatic energy integral was finite (see [BL80, Section 4, (20)–(25)]. That step was sustained by assuming that the WM-process is sufficiently close to a Gaussian process [BL80, formulae (24)–(25)]. The technique factually leads to the aforementioned Hausdorff dimension $2 - H$ of the Gauss–Weierstrass process (1.5) (see [SO1] for the case of WM).

The proximity of structure functions will factually imply the convergence of probability distributions. For this reason we call the WM-process and similar processes quasi-Gaussian. The variance of an increment has been dubbed a structure function although it does not determine the covariance configuration of a complex process. All the more, should the structure function converge still no inference could be made about the limit covariance. Therefore, we have to consider the second structure function $I(s, t)$, i.e., the covariance of the real and imaginary part of a process. These two structure functions provide a quantitative means to control the proximity of quasi-Gaussian processes.

Recall that the fBm is self-similar and the WM-process is almost self-similar. In the spirit of self-similarity and stationarity, one could expect that a quality approximation over a short time interval determines the same over an arbitrary time interval. In other words, if two process are ϵ-close over a unit interval, they should be $d(T) \cdot \epsilon$-close over an interval of length T, regardless of the position of the intervals, for some scaling factor $d(T)$. Formally, this requires to deal with the uniform convergence of the structure functions, including an estimate of $d(T)$. According to intuition, $d(T) \sim T^{2H}$. The pointwise convergence is easier to obtain but its consequence is less reliable.

1.3. Scope

Section 2 is but a compendium on complex stochastic processes with stationary increments. More information can be found in, e.g., [Yag58]. Under reasonable assumptions, the two structure functions, $D(t)$ and $I(s, t)$, determine completely its covariance configuration. The spectral representation helps to differentiate between various complex fractional motions.

In Section 3 we prove that the structure functions $D_r(t)$ and $I_r(s, t)$ of a normalized Weierstrass–Mandelbrot process $W_r(t) = \sqrt{\ln r}\, M_r(t)$ converge to the corresponding structure functions of a complex fractional Brownian motion, uniform with respect to all time moments s, t from an arbitrary time interval $[0, T]$.

In [BL80], the authors referred to the Poisson summation formula as a tool in proving the convergence of $D_r(t)$. We have found that this method alone is suitable only for $H > 1/4$. For $H \leqslant 1/4$, the approach must be augmented by an additional argument. The limit relation takes place if and only if (see Subsection 5.2.2)

$$\sum_{n \in \mathbb{Z},\, n \neq 0} \frac{e^{inx \ln(cnx)}}{|nx|^{2H+1/2}} \to 0, \qquad x \to \infty$$

This is apparently true, if $H > 1/4$. However, even the proof of the existence of the series may be a cunning problem, when $H \leqslant 1/4$. That technique might have suggested that $H = 1/4$ is a "threshold value" for the family of fBm(H)'s and related processes. The well known threshold value $H = 1/2$ partitions the class of all fBm(H)'s into persistent ($H > 1/2$), antipersistent ($H < 1/2$), and chaotic ($H = 1/2$) processes (Figs. 1 and 2). Although some other calculations that we performed also showed the need of considering somewhat special role of the parameter $H = 1/4$ but we

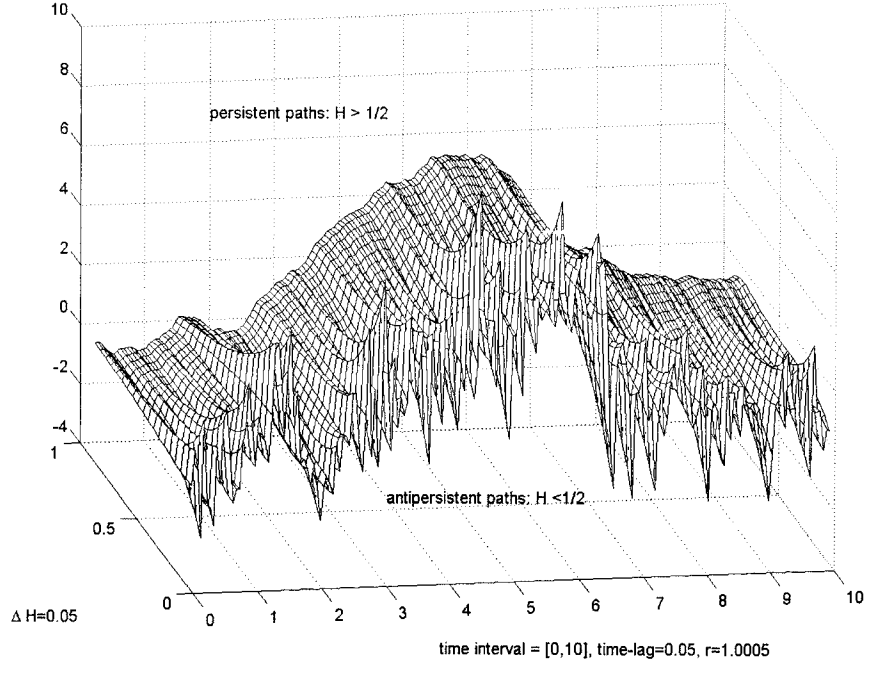

Fig. 1. Truncated series W_r, $|n| \leqslant 10,000$.

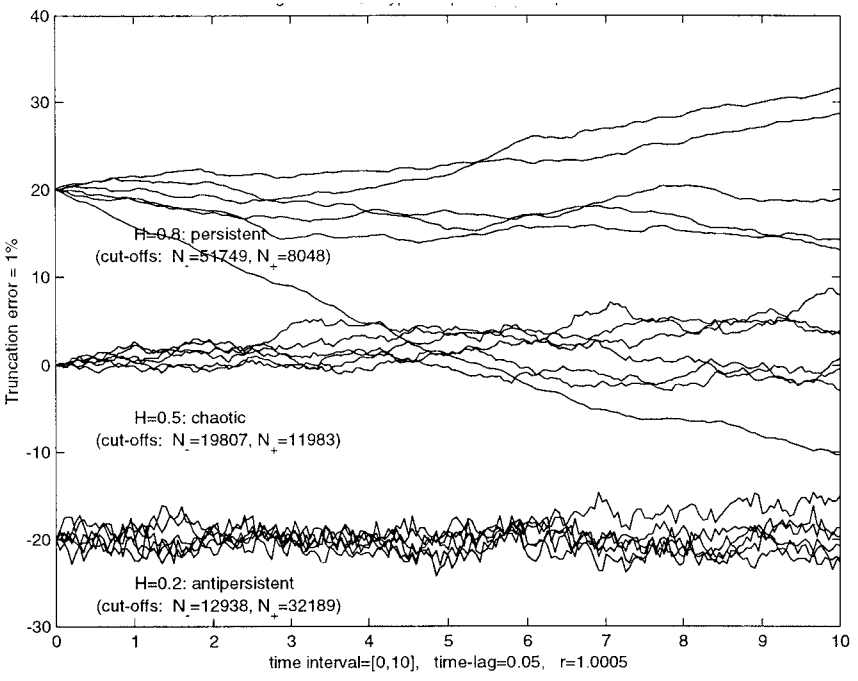

Fig. 2. The truncation of the series W_r, $-N_- \leqslant n \leqslant N_+$, minimizes the error less than 0.01.

arrived in no different conclusions that would further discriminate the category of processes. We also know of no physical reason justifying the existence of this additional special value. Our present proof is consistent with the existence of one threshold point, $H = 1/2$.

Again, the covariance structure does not determine the distribution of the process, nor the convergence of covariances implies the convergence of probability distributions. The Gaussian case is an exception. In Section 4 we introduce the class of *quasi-Gaussian* processes. Although quasi-Gaussian processes may be non-Gaussian, like the Weierstrass–Mandelbrot process is not, yet they share the important property with Gaussian processes. The convergence of the covariance function is sufficient for the convergence of finite dimensional distributions.

Further, we show that even the convergence of (finite dimensional) probability distributions of the process $\sqrt{\ln r}\, M_r(t)$ is uniform over every bounded time interval $[0, T]$.

We would like to thank the referee for his exhaustive analysis of consecutive versions of the manuscript and for making numerous illuminating remarks. Initially, the paper included a brief discussion of the numerical

simulation of the WM and related motions. Partially because of the questions asked by the referee, and partially because of the demanding sharp estimates of approximation errors, that fragment became entangled (or enriched) with extensive computation, and has exceeded the capacity of the present note. Consequently, we have partitioned the topic into a more theoretical part (the present note) and an applied part [SM00]. At the same time, we rephrased key results using a quantitative language.

A consequent research on the subject can be found in [PT00].

2. PROCESSES WITH STATIONARY INCREMENTS

2.1. Covariance Functions

We denote the mathematical expectation of a random variable X by $\mathsf{E}X$ or $\langle X\rangle$. A complex stochastic process $W(t)=X(t)+iY(t)$ is of *second order*, if $\mathsf{E}\,|W(t)|^2<\infty$, for every parameter t. In this paper we deal only with zero mean second order processes. Without loss of generality we may and do assume that our processes start at 0, $W(0)=0$ almost surely. A complex process can be seen as a two-dimensional process $(X(t), Y(t))$. Thus, its *covariance structure* is given by the three bivariate functions

$$C^R(s,t)=\mathsf{E}X(s)\,X(t), \qquad C^I(s,t)=\mathsf{E}Y(s)\,Y(t), \qquad C^M(s,t)=\mathsf{E}X(s)\,Y(t)$$

(where "C" stands for "covariance", "R" for "real", "I" for "imaginary", "M" for "mixed"). The *covariance function*

$$\begin{aligned} C(s,t)&=\langle W(s), W(t)\rangle=\mathsf{E}W(s)\,\overline{W(t)}\\ &=C^R(s,t)+C^I(s,t)+i(C^M(t,s)-C^M(s,t)) \end{aligned}$$

is insufficient to recover the complete covariance structure, in general. We say that $W(t)$ has *stationary increments*, if

$$\langle W(v)-W(t), W(u)-W(t)\rangle=\langle W(v-t), W(u-t)\rangle=C(v-t,u-t). \quad (2.1)$$

We have factually defined the stationarity in the *wide sense*. The stationarity in the *narrow sense* requires the finite dimensional distribution of the difference process to be invariant under the translation of time. The covariance structure does not determine the distribution of the processes nor the stationarity in the wide sense implies the stationarity in the narrow sense. However, Gaussian processes are exception to this rule. The examples in the introduction show two processes with stationary incre-

ments and the same covariance structure such that (1.2) is not Gaussian and (1.5) is Gaussian.

To verify the stationarity of increments of a real process $W(t)$ one needs only to check that the quantity

$$D(u) \stackrel{\text{df}}{=} \mathsf{E}\,|W(t+u)-W(t)|^2 \tag{2.2}$$

does not depend on t, and then the covariance of increments is only lag-dependent and location-independent. Indeed,

$$\langle W(u)-W(s), W(v)-W(t)\rangle = \tfrac{1}{2}\,(D(u-t)+D(s-v)-D(u-v)-D(t-s)) \tag{2.3}$$

No such relation is valid in the true complex (two-dimensional) case. Within the covariance theory, one must still verify the defining equality (2.1). The stationarity of increments of a complex valued process cannot be deduced from the same property of the real and imaginary parts, even when they have the same probability distribution. The latter condition ensures $C^R = C^I$ but leaves C^M unknown. By the same token, stationary increments of a complex processes do not secure stationary increments of the real and imaginary parts.

2.2. Spectral Theory

We refer to Subsection 5.1 for a brief on the stochastic integration. A complex valued process with stationary increments admits a *spectral representation*

$$W(t) = \int_{-\infty}^{\infty} (1-\exp\{it\omega\})\, Z(d\omega), \tag{2.4}$$

where $Z = Z_1 + iZ_2$ is an orthogonally scattered random process. The control measure (cf. Subsection 5.1) $F(d\omega)$ is often dubbed the *spectral measure* or the *spectrum* of the process. If $F(d\omega) = f(\omega)\,d\omega$, then $f(\omega)$ is called a *spectral density* or *power spectrum* of the process. The covariance function satisfies the formula

$$C(s,t) = \int_{-\infty}^{\infty} (1-\exp\{is\omega\}-\exp\{-it\omega\}+\exp\{i(s-t)\omega\})\, F(d\omega). \tag{2.5}$$

Note the representation of the real and imaginary parts of the process,

$$\begin{aligned} X(t) &= \Re W(t) = \int_{-\infty}^{\infty} (1-\cos t\omega)\, Z_1(d\omega) + \int_{-\infty}^{\infty} \sin t\omega\, Z_2(d\omega) \\ Y(t) &= \Im W(t) = -\int_{-\infty}^{\infty} \sin t\omega\, Z_1(d\omega) + \int_{-\infty}^{\infty} (1-\cos t\omega)\, Z_2(d\omega). \end{aligned} \tag{2.6}$$

Hence,

$$\begin{aligned} \Re C(s,t) &= C^R(s,t) + C^I(s,t) = \int_{-\infty}^{\infty} (1-\cos s\omega - \cos t\omega + \cos(s-t)\,\omega)\, F(d\omega), \\ \Im C(s,t) &= C^M(t,s) - C^M(s,t) = \int_{-\infty}^{\infty} (\sin(s-t)\, s\omega - \sin s\omega + \sin t\omega)\, F(d\omega). \end{aligned} \tag{2.7}$$

As we have noted above, the covariance structure (C^R, C^I, C^M) may be unrecoverable from the above formulae. In general, the integrals cannot be separated into simpler terms. The difficulties disappear if the process has the following properties

$$\left.\begin{array}{ll} (1) & Z_1 \text{ and } Z_2 \text{ are equidistributed and orthogonal} \\ (2) & \displaystyle\int_{-\infty}^{\infty} (1 \wedge \omega^2)\, F(d\omega) < \infty \end{array}\right\} \tag{2.8}$$

The first property ensures the stationarity of increments of the real processes $X(t) = \Re W(t)$ and $Y(t) = \Im W(t)$. In order to obtain a spectral representation of $X(t)$, we replace $Z(d\omega)$ (2.4) by $S(d\omega) = S_1(d\omega) + iS_2(d\omega)$, $S_1(d\omega) = (Z_1(d\omega) + Z_1(-d\omega))/2$, $S_2(d\omega) = (Z_2(d\omega) - Z_2(-d\omega))/2$. Then, $(F(d\omega) + F(-d\omega))/4$ is the spectral measure of $X(t)$ or $Y(t)$. $W(t)$ is a pure real process (i.e., $Y(t) = 0$), if $Z(-d\omega) = \overline{Z(d\omega)}$. For example, when in (1.5), $\xi_{-n} = \xi_n$ and $\xi'_{-n} = -\xi'_n$, then G_r takes only real values.

Under (2.8), the two *structure functions* fully determine the covariance structure:

$$\left.\begin{array}{l} \displaystyle D(t) \overset{\text{df}}{=} \mathsf{E}\,|W(\tau+t) - W(\tau)|^2 = 2\int_{-\infty}^{\infty} (1-\cos t\omega)\, F(d\omega) \\ \displaystyle I(s,t) = \tfrac{1}{2}\int_{-\infty}^{\infty} (\sin(s-t)\,\omega - \sin s\omega + \sin t\omega)\, F(d\omega), \qquad s > t. \end{array}\right\} \tag{2.9}$$

Indeed,

$$C^R(s,t) = C^I(s,t) = \tfrac{1}{2}\,(D(s)+D(t)-D(s-t))$$
$$C^M(s,t) = I(t,s) = -C^M(t,s). \tag{2.10}$$

Under a strengthened condition (2) in (2.8),

$$I(s) \stackrel{\text{df}}{=} \begin{cases} \displaystyle\int_{-\infty}^{\infty} \sin s\omega\, F(d\omega)/2, & \text{if, e.g., } \int_{-\infty}^{\infty} (1\wedge|\omega|)\, F(d\omega) < \infty; \\ \displaystyle\int_{-\infty}^{\infty} (\sin s\omega - s\omega)\, F(d\omega)/2, & \text{if, e.g., } \int_{-\infty}^{\infty} (|\omega|^3\wedge|\omega|)\, F(d\omega) < \infty; \end{cases} \tag{2.11}$$

the function $I(s,t)$ will take form $I(s,t) = I(s-t) - I(s) + I(t)$, $s > t$. A real valued zero-mean Gaussian process $W(t)$ with stationary increments such that $W(0) = 0$ is called a *fractional Brownian motion* with a parameter $H \in (0,1)$ (fBm or fBm(H), in short), if its structure function satisfies the formula $D(t) = \alpha\,|t|^{2H}$, for some constant $\alpha > 0$. If $H = 1/2$, then $X(t)$ is a classical Brownian motion. The fBm(H) is *self-similar* with parameter H, that is, for every real c,

$$(W(ct_1), \ldots, W(ct_n)) \stackrel{\mathscr{D}}{=} |c|^H\,(W(t_1), \ldots, W(t_n)),$$

where "$\stackrel{\mathscr{D}}{=}$" marks the identity of distributions. Most of the arguments require that $0 < H < 1$ even though one may still consider extreme trivial cases $W = W(0) = 0$, for $H = 0$, and $W(t) = W(1)\cdot t$, for $H = 1$.

We refer to a '*complex fBm*' when the real and imaginary parts of a complex process $W(t)$ are real fBms. The probability distribution of a complex fBm is not unique although it must be Gaussian. For example, let $Z = B_1 + iB_2$, where B_1, B_2 are independent classical Brownian motions. The process

$$W(t) = X(t) + iY(t) = \int_{-\infty}^{\infty} (1-\exp\{i\omega t\})\,|\omega|^{-H-1/2}\, Z(d\omega) \tag{2.12}$$

is a complex fBm such that $X(t) = \Re W(t)$ and $Y(t) = \Im W(t)$ are independent identically distributed real fBm(H)'s. Notice that $W(t)$ is time-reversible. That, is $W(t) \stackrel{\mathscr{D}}{=} W(-t)$ (or equivalently, $W(t) \stackrel{\mathscr{D}}{=} W(a-t) - W(a)$ for every a). Every real fBm has this property. In addition, the basic structure function $D(u)$ completely determines the distribution, since $I(t,s) = C^M(s,t) = 0$.

In order to show a different complex fBm, let us reduce the spectral domain $(-\infty, \infty)$ to the half-line $(0, \infty)$. The real and imaginary parts

$$W(t) = \int_0^\infty (1 - \exp\{i\omega t\})\, |\omega|^{-H-1/2}\, 2^{-1/2}(B_1(d\omega) + iB_2(d\omega)), \tag{2.13}$$

are dependent fBm(H)'s. We call this time-irreversible process the *Mandelbrot–Berry–Lewis* fBm (MBL-process, in short).

The first structure function $D(u)$ of the MBL-process equals one half of the corresponding structure function of process (2.12). More precisely,

$$D(u) = 2r_H\, |u|^{2H},$$

$$r_H = \int_0^\infty \frac{1 - \cos w}{w^{2H+1}}\, dw = \begin{cases} \dfrac{\cos(H\pi)\, \Gamma(2-2H)}{2H(1-2H)} & \text{if} \quad H \neq 1/2 \\ \pi/2 = \lim\limits_{H \to 1/2} r_H & \text{if} \quad H = 1/2 \end{cases} \tag{2.14}$$

In contrast, the second structure functions $I(s, t)$ of the MBL-process (2.13) is nontrivial,

$$I(s, t) = \frac{1}{2} \int_0^\infty (\sin(s-t)\, \omega - \sin s\omega + \sin t\omega)\, \frac{d\omega}{\omega^{2H+1}}, \qquad (s > t).$$

For $H > 1/2$ or $H < 1/2$ we can use formulae (2.11):

$$I(s, t) = I_H(s, t) = i_H((s-t)\, |s-t|^{2H} - \text{sign}(s)\, |s|^{2H} + \text{sign}(t)\, |t|^{2H}), \qquad s > t, \tag{2.15}$$

where

$$i_H = \left\{ \begin{array}{ll} \displaystyle\int_0^\infty \sin\omega\, \frac{d\omega}{\omega^{2H+1}} & \text{if} \quad H < 1/2 \\ \displaystyle\int_0^\infty (\sin\omega - \omega)\, \frac{d\omega}{\omega^{2H+1}} & \text{if} \quad H > 1/2 \end{array} \right\} = \frac{\sin(H\pi)\, \Gamma(2-2H)}{2H(1-2H)} \tag{2.16}$$

(the integrals related to r_H and i_H can be computed directly or one may consult [GR80, pp. 420-421]). By the same token, for $H = 1/2$,

$$I_{1/2}(s, t) = (t-s) \ln|s-t| - t \ln|t| + s \ln|s| \tag{2.17}$$

(by taking $\lim_{H \to 1/2} I_H(s, t)$ or using formulae from, e.g., [GR80, p. 433]). The limit approach applies also to processes (2.13), (2.12), or (1.1).

Lemma 2.1. Processes (2.13), (2.12), or (1.1) are L^2-continuous in the parameter $H \in (0, 1)$, uniformly in t from a bounded interval. More precisely, denoting any of these processes by $W_H(t)$,

$$\mathsf{E}\,|W_H(t) - W_{H'}(t)|^2 \leqslant C\,|H - H'|. \tag{2.18}$$

The constant C depends on (T, r, H, H'), with $C \to \infty$ when $\max(H, H') \to 0$ or $\min(H, H') \to 1$.

Proof. Processes (2.13) and (2.12) follow representation (2.4), with $Z = Z_H = \sqrt{f_H(\omega)}\, B(d\omega)$ and spectral measure $F_H(d\omega) = f_H(\omega)\, d\omega$. Since $|a-b|^2 \leqslant |a^2 - b^2|$ for positive a, b (below, $(a = f_H(\omega), b = f_{H'}(\omega))$, then

$$\mathsf{E}\,|W_H(t) - W_{H'}(t)|^2 = 2\int (1 - \cos(\omega t))\,\mathsf{E}\,|Z_H(d\omega) - Z_{H'}(d\omega)|^2$$

$$\leqslant 2\int (1 - \cos(\omega t))\,|F_H(d\omega) - F_{H'}(d\omega)|.$$

In the series representation (1.1) we estimate separately the sum over negative integers and the sum over nonnegative integers (with $p = \ln r$, $a = \ln t$). Let $\alpha = \min(H, H')$, $\beta = \max(H, H')$. Since,

$$\frac{d}{dH}\, p(1 - e^{-2Hp})^{-1} = \frac{2p^2 e^{-2Hp}}{(1 - e^{-2Hp})^2} \leqslant \frac{e^{2Hp}}{2H^2},$$

thus

$$p \sum_{n \geqslant 0} (1 - \cos(e^{np+a}))\,|e^{-2npH} - e^{-2npH'}|$$

$$\leqslant p \left| \frac{1}{1 - e^{-2pH}} - \frac{1}{1 - e^{-2pH'}} \right| \leqslant \frac{e^{2\beta p}}{\alpha^2}\,|H - H'|$$

Similarly, since $|(1 - \cos wt)| \leqslant (wt)^2/2$,

$$p \sum_{n \leqslant 0} (1 - \cos(e^{np+a}))\,|e^{-2npH} - e^{-2npH'}| \leqslant \frac{t^2}{2}\,\frac{e^{2(1-\alpha)p}}{(1-\beta)^2}\,|H - H'|$$

The continuous case (2.12) (or (2.13)) follows similarly, after the integral is split into one part over $[-1, 1]$ (or $[0, 1]$), and the other part, over its complement. Clearly, the constant $C = C_{H, H'}$ deteriorates, when H and H' are close to 0 or 1. ∎

A self-similar real Gaussian process with stationary increments is necessarily a fBm. Why is this definition inappropriate in the complex case? For example, let us replace in (2.13) the integrator $2^{-1/2}d(B_1+B_2)$ by dB_1. Then, the obtained Gaussian process is self-similar with exponent H, has stationary increments and even the same structure function $D(t)$. However, the second structure function $I(s, t)$ is different, so is the covariance structure. In particular, the real and imaginary parts are not fBm's for their increments are no longer stationary.

Before we state our main theorem let us notice that Gaussian processes (1.5) and the MBL-process are almost indistinguishable, up to a rescaling factor. In order to explain this statement, it suffices to observe that stochastic integral (2.12) is factually the limit in L^2 (i.e., in variance) of its approximation, cf. Subsection 5.1. For example, for $r>1$ we define the piecewise constant power spectrum by putting $r^{-n(H+1/2)}$ on the interval $(r^n, r^{n+1}]$, $n=0, \pm 1, \pm 2, \ldots$. Observe that $B_1(r_{n+1})-B_1(r_n) \stackrel{\mathscr{D}}{=} B_1(r_{n+1}-r_n) \stackrel{\mathscr{D}}{=} \sqrt{r-1}\, r^{n/2}\xi_n$, where ξ_n are independent standard Gaussian random variables. The same relation holds for B_2. Hence, this particular approximation is equidistributed with $2^{-1/2}\sqrt{r-1}\, G_r(t)$. Of course, we rely here on the convergence of the structure functions $D_r(u)$ and $I_r(u, v)$.

We state the main result (see Corollary (4.5) for a strengthened formulation).

Theorem 2.2. Let $0<H<1$. The MBL-process $W(t)$ (2.13) is the limit of the normalized WM-process $W_r(t)=\sqrt{\ln r}\, M_r(t)$, as $r \to 1$.

3. CONVERGENCE OF THE WM-STRUCTURE FUNCTIONS

The WM-process $W_r(t)=\sqrt{\ln r}\, M_r(t)$ and the MBL-process $W(t)$ satisfy (2.8). The functions $D(t)$ and $I(s, t)$ that fully determine the covariance structure of the MBL-process have been discussed in Subsection 2.11:

$$\begin{aligned} D_r(t) &= 2 \ln r \sum_{n=-\infty}^{\infty} (1-\cos(r^n t))\, r^{-2Hn}, \\ I_r(s, t) &= \tfrac{1}{2} \ln r \sum_n (\sin(r^n(s-t))-\sin(r^n s)+\sin(r^n t))\, r^{-2Hn}. \end{aligned} \tag{3.1}$$

When $H \neq 1/2$, stronger conditions (2.11) admit the simpler univariate function $I(s)$. We will use the first integral (or sum) in (2.11) for $H<1/2$, and the second integral (or sum) for $H>1/2$. We will show that, for $H \neq 1/2$, the structure functions $D_r(t)$ and $I_r(s)$ converge. The case of $H=1/2$ will follow by continuity in view of uniform estimate (2.18).

In the first line of (3.1) we put $p = \ln r$, $a = \ln t$, and $f(x) = (1 - \cos e^x)\, e^{-2Hx}$. Then

$$\frac{D_r(t)}{t^{2H}} = 2p \sum_{n \in \mathbb{Z}} f(np + a),$$

and the other structure functions follow the same pattern. Extracting the key properties, we observe that the functions

$$\Phi_p(a) \stackrel{\text{df}}{=} p \sum_{n=-\infty}^{\infty} f(np + a) \tag{3.2}$$

where $a \in \mathbb{R}$ and $p > 0$, and $f\colon \mathbb{R} \to \mathbb{R}$, are well defined, when, e.g.,

$$\begin{aligned} &|f(x)| \leqslant g(x), \qquad \text{where } g \text{ is Lebesgue integrable} \\ &\text{and monotonic outside an interval.} \end{aligned} \tag{3.3}$$

In particular, $g(x)$ and $g(-x)$ decrease to 0, eventually for large x. We will write $\langle g \rangle = \int_{-\infty}^{\infty} g(u)\, du$. A well defined $\Phi_p(a)$ is periodic with the period p, which can be seen by applying the translation $n \mapsto n + 1$ to the series.

Theorem 3.1. For a continuous function f satisfying (3.3),

$$\lim_{q \to 0} \Phi_q(a) = \langle f \rangle = \int_{-\infty}^{\infty} f(u)\, du, \qquad \text{uniform in} \quad a \in \mathbb{R}. \tag{3.4}$$

Proof. Since f is Riemann integrable, so is Φ_p, since

$$\begin{aligned} \int_0^1 \Phi_p(a + xp)\, dx &= \sum_n p \int_0^1 f(np + (a + xp))\, dx = \sum_n \int_{np}^{(n+1)p} f(u + a)\, du \\ &= \int_{-\infty}^{\infty} f(u + a)\, du = \int_{-\infty}^{\infty} f(u)\, du, \end{aligned} \tag{3.5}$$

Observe that the monotonicity and integrability of a dominant g justifies the interchange between the summation and integration in the first equality.

The family $\{\Phi_p : 1 \leqslant p \leqslant 2\}$ is uniformly equi-continuous if

$$|\Delta\Phi_p(a)| = |\Phi_p(a + \delta) - \Phi_p(a)| \leqslant \varphi(\delta), \qquad 1 \leqslant p \leqslant 2, -p \leqslant a, a + \delta < 0, \tag{3.6}$$

where $\varphi = \varphi(\delta)$ is a continuous non-decreasing function on, say, $[0, 1]$, with $\varphi(0) = 0$ and no dependence on p. Indeed, each member of that family can be written as

$$\Phi_p(a) = p \sum_{|n| \leqslant N} f(np + a) + R_N(p, a), \tag{3.7}$$

where the remainder satisfies the inequality $|R_N(p, a)| \leqslant 2 \int_{|n| > N-1} g(x)\, dx$. Hence, the property is fulfilled, because the collection of finite sums is uniformly equi-continuous and the remainder converges to 0, for $N \to \infty$, uniform in $p \in [1, 2]$ and $a \in (-p, 0]$.

Further, for every positive integer k,

$$\Phi_{p/k}(a) = \frac{1}{k} \sum_{j=0}^{k-1} \Phi_p(a + jp/k). \tag{3.8}$$

Indeed, it suffices to break the series into k sums, each taken over integers m such that $n = mk + j$, where $j = 0, \ldots, k-1$ is the remainder of division of n by k.

Let $q > 0$. Put $k = [1/q] + 1$ and $p = p_q = q([1/q] + 1)$, where $[x]$ denotes the integer part of x. Then $1 \leqslant p \leqslant 2$ and $q = p/k$. Hence, using (3.5) and (3.6),

$$\begin{aligned}
|\Delta| \stackrel{\text{df}}{=} |\langle f \rangle - \Phi_q(a)| &= \left| \int_0^1 \Phi_p(a + xp)\, dx - \Phi_{p/k}(a) \right| \\
&= \left| \int_0^1 \Phi_p(a + xp)\, dx - \frac{1}{k} \sum_{j=0}^{k-1} \Phi_p(a + jp/k) \right| \\
&\leqslant \sum_{j=0}^{k-1} \int_{j/k}^{(j+1)/k} |\Phi_p(a + xp) - \Phi_p(a + pj/k)|\, dx \\
&\leqslant \varphi(p/k) = \varphi(q).
\end{aligned}$$

This concludes the proof. ∎

The more we know about f, the quicker and easier the estimate becomes. For example, if f is differentiable and its derivative decreases rapidly, $|f'| \leqslant Ke^{-cx}$, then the use of (3.8) is unnecessary. For example, this occurs when $H > 1/2$, and $f(x) = (1 - \cos e^x) \exp\{-2Hx\}$ or $f(x) = (\sin e^x - e^x) \exp\{-2Hx\}$. One can proceed directly from (3.5). Indeed, using the fundamental theorem of calculus,

$$|\langle f \rangle - \Phi_q(a)| \leqslant \sum_n \int_0^q |f(nq + a + x) - f(nq + a)|\, dx \leqslant C_H q \to 0.$$

In the remaining cases of interest (i.e., $H < 1/2$, and $f(x) = (1 - \cos e^x)\, e^{-2Hx}$ or $f(x) = \sin e^x e^{-2Hx}$) the exponential bound increases, for $x > 0$. However, the above argument still applies for the 'negative' part of the defining series, $|\Delta^-| \leqslant C^- q$. The remainder of the argument follows by an optimization procedure, based on decomposition (3.7). It follows (cf. [SM00]), that

$$|\Delta| \leqslant C_H q^{2H}, \qquad \text{for} \quad H < 1/2, \qquad \text{and}$$

$$|\Delta| \leqslant C_{1/2} q \ln \frac{c}{q}, \qquad \text{for} \quad H = 1/2,$$

for suitable constants C_H. For further comments, see Section 5.2.

Theorem 3.2. Let $0 < H < 1$ and $r \searrow 1$. Then, the covariances $C_r^R(s, t)$, $C_r^I(s, t)$, and $C_r^M(s, t)$ of the normalized Weierstrass–Mandelbrot process $W_r(t) = \sqrt{\ln r}\, M_r(t)$ converge uniformly on intervals to the corresponding covariances $C^R(s, t)$, $C^I(s, t)$ and $C^M(s, t)$ of the MBL-process.

Proof. It suffices to see that the structure functions $D_r(t)$, for $H \in (0, 1)$, and $I_r(t)$, for $H \neq 1/2$, converge. By Theorem 3.1, for every $H \in (0, 1)$,

$$\lim_{r \to 1} \frac{D_r(t)}{t^{2H}} = 2c_f,$$

uniform in $t > 0$. Therefore $\lim_{r \to 1} D_r(t) = 2t^{2H} \int_{-\infty}^{\infty} f$, uniform on every interval. Then,

$$c_f = c_f(H) = \int_{-\infty}^{\infty} (1 - \cos e^x)\, e^{-2Hx} dx = r_H$$

In particular, the covariances of the real and imaginary parts of $W_r(t)$ converge to covariances of the real and imaginary parts of a complex fBm(H). That is ($2c_f/2 = r_H$, cf. (2.10)),

$$C_r^R(s, t) = C_r^I(s, t) \to r_H(|s|^{2H} + |t|^{2H} - |s - t|^{2H}).$$

If $H \neq 1/2$, then the simplified structure function $I(s)$ exists, cf. (2.11). Then, we apply Theorem 3.1, with $f(x) = \sin e^x e^{-2Hx}$, when $H < 1/2$ and $f(x) = (\sin e^x - e^x)\, e^{-2Hx}$, when $H \geqslant 1/2$. First, let $H \neq 1/2$. Then, (cf. (2.15))

$$\lim_{r \to 1} I_r(s, t) = \frac{a(H)}{2} (\operatorname{sign}(t - s)\, |t - s|^{2H} - \operatorname{sign}(t)\, |t|^{2H} + \operatorname{sign}(s)\, |s|^{2H}),$$

where

$$a(H) = \begin{cases} \int_{-\infty}^{\infty} \sin e^x e^{-2Hx} dx \\ \\ \int_{-\infty}^{\infty} (\sin e^x - e^x)\, e^{-2Hx} dx \end{cases} = i_H.$$

For $H = 1/2$,

$$\lim_{r \to 1} I_r(s, t) = \int_0^{\infty} (\sin(t-s)\,\omega - \sin t\omega + \sin s\omega) \frac{d\omega}{\omega^2} = I(s, t),$$

which is the imaginary part of the covariance of the complex fractional Brownian motion (2.13). This completes the proof of Theorem 3.2. ▌

Remark 3.3. 1. If $\liminf_{x \to -\infty} f(x) = c > 0$, then $\lim_p \Phi_p(a) = \infty$. Hence, for $H = 1$, $D_r(t) \to \infty$. Since $D(t) = ct^2$ for fBM(1), no convergence takes place.

2. If $H > 1/2$, the structure functions converge in a stronger sense. The derivatives of $D_r(t)$ and $I_r(s)$ converge uniformly to the derivatives of the corresponding structure functions. The functions are differentiable nowhere when $H \leqslant 1/2$.

4. CONVERGENCE OF DISTRIBUTIONS

We introduce a category of stochastic processes whose distribution is determined by the covariance structure, to some extent. These processes do not have to be Gaussian yet the aforementioned property is an important feature of a Gaussian process.

Consider a sequence $\zeta = (\zeta_n)$ of random planar vectors $\zeta_n = (\xi_n, \eta_n)$, $n \in \mathbb{Z}$, such that

$$\left.\begin{array}{l} \zeta_n \text{ are independent and symmetric;} \\ \mathsf{E}\xi_n^2 = \mathsf{E}\eta_n^2; \\ \text{for every } n,\ \xi_n \text{ and } \eta_n \text{ are uncorrelated.} \end{array}\right\} \tag{4.1}$$

Without loss of generality we may and do assume that $\mathsf{E}\xi_n^2 = \mathsf{E}\eta_n^2 = 1$. For a sequence $z = (z_n)$ of vectors $z_n = (x_n, y_n)$ such that

$$\|z\|_2^2 \stackrel{\text{df}}{=} \sum_n |z_n|^2 = \sum_n (x_n^2 + y_n^2) < \infty,$$

the random function

$$\langle z, \zeta \rangle \stackrel{\text{df}}{=} \sum_{n \in \mathbb{Z}} (x_n \xi_n + y_n \eta_n) \tag{4.2}$$

is well defined under (4.1). A complex random variable $W = X + iX'$ is called *quasi-Gaussian*, if $X = \langle z, \zeta \rangle$ and $X' = \langle z', \zeta \rangle$, for some z, z', sequences of pairs of real numbers. We dub a complex stochastic process $W(a)$ *quasi-Gaussian*, if each $W(a)$ is quasi-Gaussian, and only coeffcients (z_n) depend on the parameter a while the random sequence ζ is parameter-free.

The non-Gaussian WM-process is quasi-Gaussian. More examples can be constructed easily by replacing the random variables $e^{i2\phi_n}$ in (1.1) by random variables ζ_n, satisfying (4.1).

A quasi-Gaussian process (or variable) can be immediately turned into a Gaussian process by replacing the vector (ξ_n, η_n) by a pair of independent standard Gaussian random variables, for every n. For example, process (1.5) is a Gaussian counterpart of the WM-process.

A process with spectral representation (2.4) with a Gaussian spectral random measure Z is quasi-Gaussian. We normalize terms in (2.6). First, we partition the real line into intervals $[n, n+1)$. Then we can use independent pairs of independent standard Gaussian random variables. Also, e.g.,

$$x_n^2 = \int_n^{n+1} (1 - \cos t\omega)^2 \, \mathsf{E} Z_1^2(d\omega) + \int_n^{n+1} \sin^2 t\omega \mathsf{E} Z_2^2(d\omega),$$

and y_n follows the similar pattern corresponding to the second line in (2.6).

We will estimate the Fourier transform of a quasi-Gaussian process. Since $\langle z, \zeta \rangle$ is a linear stochastic process in the parameter $z = (z_n)$, hence $\mathsf{E} \exp\{i\langle z, \zeta \rangle\}$ is general enough. Below and elsewhere, $\|z\|_\infty \stackrel{\text{df}}{=} \sup_n |z_n|$.

Proposition 4.1. Assume (4.1). Then, for $k \geqslant 2$ and $\|z\|_\infty \leqslant 1$,

$$\left| \mathsf{E} \exp\{i\langle z, \zeta \rangle\} - \exp\left\{ -\frac{\|z\|_2^2}{2} \right\} \right| \leqslant \frac{k^4}{24} \|z\|_\infty^2 \|z\|_2^2 + \frac{\epsilon_k(z)}{2} \|z\|_2^2, \tag{4.3}$$

where

$$\epsilon_k(z) \stackrel{\text{df}}{=} \sup_n \mathsf{E}[|\zeta_n|^2; |\zeta_n| > k].$$

Proof. The following inequality follows by a routine calculus

$$-v^2 \leqslant \ln(1-v+u)+v \leqslant u, \qquad \text{whenever} \quad 0 \leqslant u \leqslant v < \tfrac{1}{2}. \tag{4.4}$$

Let X be a random variable such that $\mathsf{E}\,|X|^2 \leqslant 1$. Put

$$u = \mathsf{E}\left(\cos X - 1 + \frac{X^2}{2}\right) > 0 \tag{4.5}$$

(as $\cos x - 1 + x^2/2 > 0$), and $v = \mathsf{E}X^2/2$. Then, inequality (4.4) implies that

$$-\frac{(\mathsf{E}\,|X|^2)^2}{4} \leqslant \ln \mathsf{E}\cos X + \frac{\mathsf{E}X^2}{2} \leqslant \mathsf{E}\left(\cos X - 1 + \frac{X^2}{2}\right) \tag{4.6}$$

When X is symmetric, then $\mathsf{E}e^{iX} = \mathsf{E}\cos(X)$. Let $\|z_n\|_2^2 = \mathsf{E}\langle z_n, \zeta_n\rangle^2 \leqslant 1$. Then

$$\begin{aligned}
&\left|\mathsf{E}\exp\left\{i\sum_n \langle z_n, \zeta_n\rangle\right\} - \exp\{-\|z\|^2/2\}\right| \\
&\quad = \left|\exp\left\{\sum_n \ln \mathsf{E}\cos\langle z_n, \zeta_n\rangle\right\} - \exp\left\{-\sum_n (x_n^2+y_n^2)/2\right\}\right| \\
&\quad \leqslant \sum_n \left|\ln \mathsf{E}\cos\langle z_n, \zeta_n\rangle + \sum_n (x_n^2+y_n^2)/2\right|.
\end{aligned}$$

We will find an upper bound of the term under the absolute value. By (4.6),

$$\ln \mathsf{E}\cos\langle z_n, \zeta_n\rangle + \frac{\mathsf{E}\langle z_n, \zeta_n\rangle^2}{2} \leqslant \mathsf{E}(\cos\langle z_n, \zeta_n\rangle - 1 + \langle z_n, \zeta_n\rangle^2/2). \tag{4.7}$$

Now, consider two inequalities

$$0 \leqslant \cos x - 1 + \frac{x^2}{2} \leqslant \frac{x^4}{24}. \qquad \text{and} \quad \cos x - 1 + \frac{x^2}{2} \leqslant \frac{x^2}{2}. \tag{4.8}$$

Then, we split the expectation in the right hand side of (4.7) into the sum of two terms, one being the expectation, restricted to the event $\{|\zeta_n| \leqslant k\}$, and the other, restricted to its complement, $\{|\zeta_n| > k\}$. For the first term, we apply the first inequality of (4.8):

$$\begin{aligned}
\mathsf{E}[\cos\langle z_n, \zeta_n\rangle - 1 + \langle z_n, \zeta_n\rangle^2/2;\, |\zeta_n| \leqslant k] &\leqslant \frac{1}{24}\,\mathsf{E}[(x_n\xi_n + y_n\eta_n)^4;\, |\zeta_n| \leqslant k] \\
&\leqslant \frac{k^4}{24}\,(x_n^2+y_n^2)^2
\end{aligned} \tag{4.9}$$

When $k \geqslant 2$, then $k^4/24 > 1/4$ (1/4 is the constant in the lower bound in formula (4.6)). This leads to the first term on the right hand side of (4.3). Now, let us apply the second inequality (4.8) to the expectation, restricted to $\{|\zeta_n| > k\}$:

$$\mathsf{E}[\cos\langle z_n, \zeta_n\rangle) - 1 + \langle z_n, \zeta_n\rangle^2/2; |\zeta_n| > k] \leqslant \tfrac{1}{2}|z_n|^2\, \mathsf{E}[|\zeta_n|^2; |\zeta_n| > k].$$

The summation over n yields the second term on the right hand side of (4.3). ▮

The assumption of identical distribution below may be weakened by the requirement of the uniform integrability of variables $|\zeta_n|^2$, i.e., by the condition $\lim_{k\to\infty} \epsilon_k(z) = 0$.

Corollary 4.2. Let the assumptions of Proposition 4.1 be satisfied. Let $z_n = z_n(p)$ depend on the variable $p \in (0, 1]$ such that $\|z(p)\|_\infty \to 0$, and $a = \lim_{p\to 0} \|z(p)\|_2$ exists. If ζ_n are equidistributed, then

$$\lim_p \mathsf{E} \exp\{i\langle z(p), \zeta\rangle\} = \exp\{-a^2/2\} \tag{4.10}$$

Proof. For $z = z(p)$, assume that $\|z\|_\infty \leqslant 1$, without loss of generality. In addition to the triangle inequality, we use below the inequality from Proposition 4.1, and the estimate $|e^{-x} - e^{-y}| \leqslant |x - y|$, $x, y \geqslant 0$. Thus,

$$\begin{aligned}
&|\mathsf{E} \exp\{i\langle z, \zeta\rangle\} - \exp\{-a^2/2\}| \\
&\quad \leqslant |\mathsf{E} \exp\{i\langle z, \zeta\rangle\} - \exp\{-\|z\|_2^2/2\}| + |\exp\{-\|z\|_2^2/2\} - \exp\{-a^2/2\}| \\
&\quad \leqslant \frac{k^4}{24}\|z\|_\infty^2 \|z\|_2^2 + \epsilon_k \|z\|_2^2 + (\|z\|_2^2 - a^2)/2.
\end{aligned}$$

Since ζ_n have the same distribution, then $\epsilon_k \to 0$. Thus, we can choose k large enough to make the second term on the right hand side of the latter inequality arbitrarily small. Then, we let p converge, leading to null limits of the first and third terms. Therefore, the difference on the left hand side becomes as small as desired. ▮

Roughly speaking, the convergence of variances will imply the convergence of distributions, which extends the important feature of Gaussian processes. By rewriting the inequality in Corollary 4.2, we arrive at the following estimate of the rate of convergence.

Corollary 4.3. Let the assumptions of Proposition 4.2 be satisfied. Assume additionally that $\|\zeta\|_\infty \leqslant 1$. Then

$$|\mathsf{E}\exp\{i\langle z, \zeta\rangle\} - \exp\{-a^2/2\}| \leqslant \|z\|_\infty^2 \, \|z\|_2^2/4 + (\|z\|_2^2 - a^2)/2.$$

In particular, the estimate holds for the WM-process.

Proof. If ζ_n are bounded random variables, $|\zeta_n| \leqslant K$, then we can simplify the right hand side of (4.3), since $\epsilon_k = 0$ for sufficiently large k. In Proposition 4.1 we required $k \geqslant 2$ for merely aesthetic reasons. As explained in the last paragraph of the proof of that proposition, if $|\zeta_n| \leqslant 1$ (e.g, for the Weierstrass–Mandelbrot process), then $\max(k^4/24, 1/4)$ is a smaller constant. Since $\epsilon_k = 0$, we may and do improve the constant by choosing $k = 1$. ▌

The limits of covariance functions are covariance functions again. In particular, there is a complex Gaussian process $G(t) = X(t) + iX'(t)$ with these covariances.

Theorem 4.4. Let $\{W_p(t) = X_p(t) + iX'_p(t), p > 0\}$ be a family of quasi-Gaussian stochastic processes. Suppose that the covariance functions converge pointwise, for $p \to 0$, and choose a complex Gaussian process $G(t) = X(t) + iX'(t)$ with the limit covariance functions. That is, suppose that for every s, t, as $p \to 0$,

$$\begin{aligned} C_p^R(s, t) &= \mathsf{E}X_p(s)\, X_p(t) \to C^R(s, t) = \mathsf{E}X(s)\, X(t) \\ C_p^I(s, t) &= \mathsf{E}X'_p(s)\, X'_p(t) \to C^I(s, t) = \mathsf{E}X'(s)\, X'(t) \\ C_p^M(s, t) &= \mathsf{E}X_p(s)\, X'_p(t) \to C^M(s, t) = \mathsf{E}X(s)\, X'(t). \end{aligned} \tag{4.11}$$

If, for every t,

$$\begin{aligned} \|z_p(t)\|_\infty^2 &= \sup_n |z_{p,n}(t)|^2 = \sup_n (|x_{p,n}(t)|^2 + |y_{p,n}(t)|^2) \to 0, \\ \|z'_p(t)\|_\infty^2 &= \sup_n (|x'_{p,n}(t)|^2 + |y'_{p,n}(t)|^2) \to 0, \end{aligned} \tag{4.12}$$

then the finite dimensional distributions of the process $W_p(t)$ converge weakly to the corresponding finite dimensional distributions of $G(t)$.

Proof. Let $t_1, \ldots, t_d$ be any sequence of times. Denote $W_{p,k} = W_p(t_k)$, $X_{p,k} = X_p(t_k)$, $X'_{p,k} = X'_p(t_k)$. Then,

$$W_{p,k} = X_{p,k} + iX'_{p,k}, \qquad X_{p,k} = \sum_n \langle z_{p,kn}, \zeta_n\rangle,$$

$$X'_{p,k} = \sum_n \langle z'_{p,kn}, \zeta_n\rangle, \qquad k = 1, \ldots, d,$$

where $z_{p,kn} = (x_{p,kn}, y_{p,kn})$, $z'_{p,kn} = (x'_{p,kn}, y'_{p,kn})$ are pairs of real numbers. The distribution of the $(2d)$-dimensional vector $(W_{p,1}, \ldots, W_{p,d})$ is determined by the distributions of all real linear forms

$$\sum_{k=1}^{d} (\alpha_k X_{p,k} + \alpha'_k X'_{p,k}) = \sum_n (x_{p,n}\xi_n + y_{p,n}\eta_n), \tag{4.13}$$

with

$$x_{p,n} = \sum_k (\alpha_k x_{p,kn} + \alpha'_k x'_{p,kn}), \qquad y_{p,n} = \sum_k (\alpha_k y_{p,kn} + \alpha'_k y'_{p,kn}). \tag{4.14}$$

If we denote covariances of the vector $(X_{p,1}, \ldots, X_{p,d}, X'_{p,1}, \ldots, X'_{p,d})$ by

$$C_p^R(j,k) = \sum_n (x_{p,jn}x_{p,kn} + y_{p,jn}y_{p,kn})$$

$$C_p^I(j,k) = \sum_n (x'_{p,jn}x'_{p,kn} + y'_{p,jn}y'_{p,kn})$$

$$C_p^M(j,k) = \sum_n (x_{p,jn}x'_{p,kn} + y_{p,jn}y'_{p,kn}),$$

then

$$\sum_n (x_{p,n}^2 + y_{p,n}^2) = \sum_{j,k} \alpha_j\alpha_k C_p^R(j,k) + \sum_{j,k} \alpha'_j\alpha'_k C_p^I(j,k) + 2\sum_{j,k} \alpha_j\alpha'_k C_p^M(j,k). \tag{4.15}$$

Like in Proposition 4.1, the characteristic function of linear form (4.13) depends on the quantity on the left hand side, and by assumption it converges to the characteristic function of $\sum_{k=1}^{d} (\alpha_k X(t_k) + \alpha'_k X'(t_k))$. Therefore, the convergence of covariances implies the convergence of finite dimensional distributions. ▌

Theorem 4.4 and Theorem 3.2 lead to a quantitative version of Theorem 2.2 that provides an estimate of the approximation error. Let us denote the supremum norms by $\|D\|_{T,\infty} = \sup_{0 \leqslant t \leqslant T} |D(t)|$ and $\|I\|_{T,\infty} = \sup_{0 \leqslant s,t \leqslant T} |I(s,t)|$, with or without subscript "r". For $H < 1/2$ or $H > 1/2$ one could employ the univariate functions $I(t)$ and $I_r(t)$ (cf. comments following (2.11)). More intricate estimates are given in [SM00, Section 2].

Corollary 4.5. The limit relation Theorem 2.2 is valid in the sense of the convergence of finite dimensional distributions, uniform in every interval $[0, T]$.

$$\left|\mathsf{E}\exp\left\{i\sum_k(\alpha_k X_r(t_k)+\alpha'_k Y_r(t_k)\right\}-\mathsf{E}\exp\left\{i\sum_k(\alpha_k X(t_k)+\alpha'_k Y(t_k)\right\}\right|$$
$$\leqslant C_{T,A}(\ln r+\|D_r-D\|_{T,\infty}+\|I_r-I\|_{T,\infty}), \tag{4.16}$$

where $\|\alpha\|_1 \leqslant A$, $\|\alpha'\|_1 \leqslant A$ ($\|\alpha\|_1 = \sum_k |\alpha_k|$), and a constant $C_{T,A}$ depends only on (T, A).

Proof. It suffices to recognize the suitable quantities that appear in Theorem 4.4. First, we return to denoting the imaginary parts of the processes by the capital letter Y instead of X'. Next, we have $p = \ln r \to 0$. Then, $\zeta_n = (\xi_n, \eta_n)$, where $\xi_n = \cos\phi_n$ and $\eta_n = \sin\phi_n$. Further,

$$x_{p,kn} = \sqrt{p}\,(1-\cos(e^{pn}t_k))\,e^{-Hpn} = y'_{p,kn},$$
$$y_{p,kn} = \sqrt{p}\sin(e^{pn}t_k)\,e^{-Hpn} = -x'_{p,kn}.$$

Given coefficients (α_k), (α'_k), we now obtain $x_{p,n}$ and $y_{p,n}$ as in (4.14), and, consequently, vectors $z_{p,n} = (x_{p,n}, y_{p,n})$.

Let us turn to the inequality from Corollary 4.3 that involves $\|z_p\|_\infty$, $\|z_p\|_2^2$, with z_p replacing z, and

$$a^2 = \mathsf{E}\left|\sum_k(\alpha_k X(t_k)+\alpha'_k Y(t_k))\right|^2.$$

Assume that $t_k \leqslant T$ and recall (2.10). Then, (4.15) implies

$$\|z_p\|_2^2 - a^2 \leqslant 3A^2\|D_r-D\|_{T,\infty}+2A^2\|I_r-I\|_{T,\infty}.$$

An upper bound of $\|z_p\|_2^2$ is obtained similarly. To control the supremum norm $\|z_p\|_\infty^2$ we may use the inequality

$$|z_{p,n}|^2 \leqslant 2\left(\sum_k(\alpha_k^2+\alpha_k'^2)^{1/2}\right)^2(2\vee T^2)\,p \leqslant 8A^2(2\vee T^2)\,p \to 0.$$

Finally, we merge all obtained constants into a vague but lucid constant $C_{T,A}$. ∎

5. APPENDIX

5.1. Stochastic Integral

Let $Z(t)$ be a complex valued stochastic process. For an interval $A = (a, b]$, we define the *random measure* $Z(A) \stackrel{\text{df}}{=} Z(b) - Z(a)$. Let ϕ be a piecewise constant function, i.e., $\phi = \sum_{k=1}^{n} z_k \mathbb{1}_{A_k}$, where z_k are complex numbers, A_k are disjoint intervals, and $\mathbb{1}_A$ denotes the indicator 0-1 function. The rudimentary *stochastic integral* follows the usual definition, $\int \phi \, dZ = \sum_k z_k Z(A_k)$. Of course, one wants to extend the integral beyond the class of simple functions. In general, a calculus-style extension fails because typical trajectories of processes are highly irregular. For example, trajectories of a Brownian motion have unbounded variation which makes the Riemannian approach useless. While the use of generalized functions (i.e., of distributions) is a promising choice, the processes in our purview allow a direct and simple probabilistic treatment.

Let a second order process $Z(t)$ be *orthogonally scattered*, i.e., for disjoint intervals A and B, the random variables $Z(A)$ and $Z(B)$, defined above, are orthogonal (i.e., uncorrelated). Without loss of generality we may assume that its mean is zero, $\mathsf{E}Z(t) = 0$. For an interval A, $F(A) \stackrel{\text{df}}{=} \mathsf{E}\,|Z(A)|^2$ defines a deterministic *control measure* of the random measure Z and leads to the isometry

$$\mathsf{E}\left|\int \phi dZ\right|^2 = \int |\phi|^2 \, dF, \qquad \text{where } \phi \text{ is piecewise constant.} \tag{5.1}$$

Piecewise-constant functions with bounded support form a dense set in the Hilbert space

$$L^2(F) = \left\{ \phi : \int |\phi|^2 \, dF < \infty \right\}.$$

Hence, the stochastic integral $\int \phi \, dZ$ can be extended to all integrands from $L^2(F)$. That is, for every square integrable function ϕ, the random variable

$$\int \phi \, dZ = \int_{-\infty}^{\infty} \phi(\omega) \, Z(d\omega)$$

with finite variance is well defined. In particular, the random measure $Z(J)$, defined originally for intervals J only, now makes sense for a greater variety of sets from the real line through the formula $Z(A) = \int \mathbb{1}_A dZ$, if only $F(A)$ exists and is finite. Also, by considering an increasing sequence

r_n, $n \in \mathbb{Z}$, and a function ϕ that is equal to a constant z_n on the interval $(r_n, r_{n+1}]$, $n \in \mathbb{N}$, we obtain the formula

$$\int_{-\infty}^{\infty} \phi(\omega)\, Z(d\omega) = \sum_n z_n(Z(r_{n+1}) - Z(r_n)), \qquad \text{if} \quad \sum_n |z_n|^2 F([r_{n+1}, r_n)) < \infty. \tag{5.2}$$

Increments of a complex process and of its real and imaginary parts are not necessary orthogonal at the same time.

5.2. Theorem 3.1

5.2.1. Strength of Assumptions

Some properties of functions Φ_p are purely algebraic, and they hold once these functions are well defined (for example, the periodicity and (3.8)). In contrast, components of the proof of Theorem 3.1 rely on the assumed properties of f, the continuity and condition (3.3). For example, both properties are used to justify the interchange between the series and the integral at the first equality in (3.5), and, a fortiori, make Φ integrable (in the same sense f was, Riemann's or Lebesgue's). Even if we knew that Φ_p was Riemann integrable, then formula (3.8) would merely suggest that, for every a,

$$\Phi_p(a) \sim \langle f \rangle, \qquad \text{when} \quad p \to 0. \tag{5.3}$$

However, without the aforementioned assumptions, the only legitimate and rigorous approach to the limit relation " $\sim$ " would be to understand it as a rather crude type of convergence along a subsequence, and even this subsequence may depend on a.

Is a monotonic dominant necessary? All the arguments above are rigorous and obvious, if, for example, a Riemann integrable function f has a bounded support, that is, when f vanishes outside an interval. Indeed, the series (3.2) and the subsequent series become then just finite sums. Yet, even then (5.3) will bear the impaired meaning. For functions of true interest, with unbounded support, not even their integrability and continuity together will make the argument viable. First of all, series (3.2) may diverge, rescinding any farther discussion. We leave the design of examples to the reader.

Assumptions of Theorem 3.1 embark a compromise between the distinctive form of our functions and a desire to find the weakest condition leading to (3.4).

5.2.2. Poisson Summation Formula

The formula is one of the celebrated tools used in Fourier analysis and related areas (which includes the subject of this paper, too). Berry and Lewis [BL80] mentioned it as a way of showing that the normalized WM-structure function becomes the structure function of fBm in the limit. In the introduction we mentioned a shortcoming of this approach. Because of its popularity and extensive references (e.g., see [AB85]) to that paper, we feel that we owe an explanation. However, complete computations are rather cumbersome. Therefore, we will display only the key steps with sufficient references to sources.

Let F be a continuous function on the real line. The Poisson summation formula states that

$$\sum_m F(m) = \sum_m \hat{F}(m),$$

where $\hat{F}(m)$ is the Fourier coefficient $\hat{F}(m) = \int_{-\infty}^{\infty} F(x)\, e^{-2\pi i x m}\, dx$. The idea of the argument lies in showing that

$$\lim_{p \to 0} \sum_{m \neq 0} \hat{F}_p(m) = 0. \tag{5.4}$$

This would imply that

$$\lim_{p \to 0} \sum_m F_p(m) = \lim_{p \to 0} \hat{F}_p(0) = \int_{-\infty}^{\infty} (1 - \cos(e^x))\, e^{-2Hx}\, dx$$

$$= \int_0^{\infty} (1 - \cos(u))\, u^{-2H-1}\, du,$$

which agrees with the contents of (3.4), for $f(x) = (1 - \cos e^x)\, e^{-2Hx}$. Here, $a = \ln t$ and $p = \ln r$. The treatment of Fourier coefficients

$$\hat{F}_p(m) = p \int_{-\infty}^{\infty} (1 - \cos(e^{px+a}))\, e^{-2H(px+a)} e^{-2\pi i x m}\, dx \tag{5.5}$$

can be divided into three steps. First, $\hat{F}_p(m)$ can be expressed in terms of Gamma and trigonometric functions of complex variable (using standard integration techniques),

$$\hat{F}_p(m) = \frac{\Gamma(z_m) \sin \pi z_m / 2}{2H + 2\pi i m / p}\, e^{2\pi m i a / p}, \tag{5.6}$$

where $z_m = -2H + 1 - 2\pi i m/p$. Three cases, $H < 1/2$, $H = 1/2$, and $H > 1/2$, are slightly different. For example, for $H > 1/2$, the function Γ must be extended to the part of the complex plain $\{\Re z < 0\}$. For details we refer to textbooks or guides on complex calculus. The tables of series and integrals with theoretical briefs [GR80] (formula 8.312.8, p. 934) may also serve the present purpose.

The next step deals with the asymptotic behavior of the obtained expression. What for real Gamma function is known as Stirling's formula, now takes the form (e.g., referring again to [GR80], p. 937, 8.327 or 8.328.1, when $H \neq 1/2$, and 8.332.1 when $H = 1/2$ i.e., $\Re z_m = 0$),

$$|\Gamma(z) \sin \pi z/2| \approx \sqrt{\pi/2}\, |\Im z|^{\Re z - 1/2}, \qquad \text{as} \quad |\Im z| \to \infty. \tag{5.7}$$

Therefore, for $p \to 0$,

$$|\hat{F}_p(m)| \sim \frac{|2\pi m/p|^{-2H+1/2}}{|2H + 2\pi i m/p|} = \frac{p^{2H+1/2}}{|2Hp + m| m^{2H-1/2}} \sim \frac{p^{2H+1/2}}{m^{2H+1/2}}. \tag{5.8}$$

The argument is finished if $H > 1/4$. That is, (5.4) holds, because the series $\sum_{m \neq 0} \hat{F}_p(m)$ converges absolutely and uniformly.

The third step is needed to handle the case $H \leqslant 1/4$, because the domination (Weierstrass') criterion is now useless. In order to continue, $\hat{F}_p(m)$ would have to be exposed to a more subtle procedure. One could replace terms by new simpler terms, as long as the differences form uniformly and absolutely convergent series. Details are quite onerous, but one can show that (5.4) holds, if and only if, the following series is convergent (conditionally), and its limit is 0, as $x = 1/p \to \infty$.

$$\sum_{m \in \mathbb{Z},\, m \neq 0} \frac{e^{imx \ln(cmx)}}{|mx|^{2H+1/2}} \to 0, \qquad x \to \infty,$$

for a suitable constant c. It is quite challenging to prove the convergence directly.

Of course, we do know that the rest of the argument can be completed somehow, for Theorem 3.1 has been proved. However, we feel reluctant to accept that (5.4) follows from the Poisson summation formula entirely. Nonetheless, the very method induces a special phase transition while the parameter H passes through the point $H = 1/4$ on the parameter scale $[0, 1]$. We do not know whether the special point is tangible or it is only superficial.

5.2.3. Pointwise Convergence

The referee pointed that our argument did not reveal immediately the very integral nature of structure functions. For example, $D_r(t)$ (3.1) can be rewritten, replacing $\ln r \approx (r-1)/r$, as follows

$$\begin{aligned} D_r(t) &\approx \frac{r-1}{r} \sum_{n=-\infty}^{\infty} (1-\cos(r^n t))\, r^{-2Hn} \\ &= \sum_{n=-\infty}^{\infty} (1-\cos(r^n t))\, r^{-2Hn}(r^n - r^{n-1})\, r^{-n} \\ &= \sum_{n=-\infty}^{\infty} \int_{r^{n-1}}^{r^n} (1-\cos([r^n]\, t))[r^n]^{-2H-1}\, dx, \end{aligned}$$

which is an approximation of the integral

$$\int_0^{\infty} (1-\cos(xt))\, x^{-2H-1}\, dx.$$

We recognize the Lebesgue integral of a step function, constant on intervals $(r^{n-1}, r^n]$, which converges pointwise to the integrand, as $r \to 1$. The dominant, equal to $x^{-2H+1}t^2/2$, for $x \in (0, 1)$, and equal to x^{-2H-1}, for $x \in [1, \infty)$, is integrable on $(0, \infty)$. Hence, by the dominated convergence theorem

$$\lim_{r \to 1} D_r(t) = \int_0^{\infty} (1-\cos(xt))\, x^{-2H-1}\, dx,$$

for every $t \leqslant T$. One can treat the other structure function $I_r(t)$ similarly (when $H \neq 1/2$, for the case $H = 1/2$ requires a special approach since the variables in $I_r(s, t)$ cannot be separated).

This immediate conclusion of the convergence for each separate t provides no clue regarding the uniform convergence with respect to t or the rate of convergence. In the absence of these factors, the quality of potential simulation of trajectories would suffer. To achieve the uniform convergence and to control the rate of convergence, one needs an estimate of the variability of $D_r(t)$, subordinate to the same property of f. In other words, either the proof of Theorem 3.1 would have to be emulated or an additional argument provided.

5.3. Probability Metrics and Uniform Convergence

Consider first the ensemble of probability laws of real random variables. There are many ways of giving a quantitative meaning to vague

statements like '*the probability laws of X and Y are close*'. Such ways originated in works of Lévy and Kolmogorov. An explicit theory of the so-called metrics in probability spaces has been developed by Zolotarev, and extended further by many authors. The monograph [Rac91] may serve as an exhaustive guide to this vast subject. A classical quantity measuring the distance between two probability laws, say, in terms of cumulative distribution functions, is the Gnedenko–Kolmogorov metric

$$\mathsf{abs}(X, Y) = \sup_c |\mathsf{P}(X \leqslant c) - \mathsf{P}(Y \leqslant c)|.$$

For graphical reasons, the abbreviation $\mathsf{abs}(X, Y)$ is often used in place of the proper symbol $\mathsf{abs}(\mathscr{L}(X), \mathscr{L}(Y))$. Clearly, $\mathsf{abs}(X, Y) = 0$ implies the equality of probability laws, not of random variables.

This metric is factually too strong in areas where the convergence of distributions (like in the central limit theorem) is of interest. Among a plethora of metrics, quantifying the convergence of probability of laws (again, cf. [Rac91]), the difference of characteristic functions (i.e., of Fourier transforms)

$$|\mathsf{E}e^{iaX} - \mathsf{E}e^{iaY}|$$

is often useful. Of course, the quantity, depending on a solitary a, is not a metric itself, for a must run through all real multipliers. One needs to consider a family of *semi-metrics* which may be turned to a metric. Lévy's topology of uniform convergence of characteristic functions on compact intervals (cf. also [Lu70]) is equivalent to the convergence of probability laws. A typical procedure, beginning with

$$d_k(X, Y) = \sup_{|a| \leqslant k} |\mathsf{E}e^{iaX} - \mathsf{E}e^{iaY}|, \text{ leads to a true metric}$$

$$d(X, Y) = \sum_k \frac{d_k(X, Y)}{2^k}. \tag{5.9}$$

The extension of this quantitative approach to random vectors in a n-dimensional space is straightforward. It suffices to interpret a and X above as vectors, aX as the scalar product, and $|a|$ as the Euclidean distance from the origin.

In the ensemble of probability laws $\mathscr{L}(W)$ of complex or even real random processes $W = W(t)$ on the interval $[0, T]$ the situation is more complicated. First, the term 'convergence' refers to a much greater number of non-equivalent definitions. For example, a process with continuous trajectories may be seen as a random variable taking values in the metric (Banach, in fact) space $C[0, T]$. As such, its distribution generates a probability measure in this space. Alternatively, sequences of times $t_1, t_2, \ldots, t_n$

of various length plunge the selection $(X(t_1), \ldots, X(t_n))$ into Euclidean spaces. In particular, '*the convergence of finite dimensional distributions*' is essentially weaker then '*the convergence of distributions*'. The latter usually requires a tremendous machinery of the general theory of stochastic processes and induces a great deal of measure-theoretic problems (cf., e.g., [JS87]).

Even the convergence of finite dimensional distributions may show varying degrees of strength. The quoted qualitative definition, as in the vicinity of (1.4), refers to various choices of moments $t_1, \ldots, t_n$. We may expect a priori a weakening of convergence as time elapses. Even the controlled growth of the variance (exactly of order t^{2H} for the limit MBL-process, and approximately of order t^{2H} for the WM-process) does not govern the deviation between distributions, in general, yet it may suggest a scaling factor (wishfully, of the given order). We factually prove that the obtained properties hold uniformly with respect to time.

To measure the distance between the finite dimensional distributions of processes $W(t) = X(t) + iY(t)$ and $W'(t) = X'(t) + iY'(t)$, we can design a uniform analog of (5.9), $d_A(W, W') \stackrel{\text{df}}{=}$

$$\sup \left| \mathsf{E} \exp \left\{ i \sum_k (a_k X(t_k) + b_k Y(t_k) \right\} - \mathsf{E} \exp \left\{ i \sum_k (a_k X'(t_k) + b_k Y'(t_k) \right\} \right|, \tag{5.10}$$

where the supremum is taken over n, $k = 1, \ldots, n$, n are arbitrary integers, $z_k = a_k + ib_k$, with $\sum_k |a_k| \leqslant A$, $\sum_k |b_k| \leqslant A$, and $|t|_\infty = |(t_1, \ldots, t_n)|_\infty = \max_k t_k \leqslant T$. Of course, (5.10) may be inconvenient, when used directly. Therefore, we take advantage of an efficient estimate (see Proposition 4.1 and its corollaries).

This brings us to the class of quasi-Gaussian processes. Recall that the possibility of a direct transformation into a Gaussian process was one of reasons behind this name. For the sake of clarity, let us confine to the processes with bounded summands (with the unit bound, like in the WM-process), and turn our attention to the upper bound in Corollary 4.3, and its rephrase, (4.16). The upper bound contains two terms. We notice that the first term controls the formal switch to a Gaussian process, while the second term measures the deviation of variances. The first term results from an upper bound of the distance, by means of metric d_A (5.10), between the original process and its formal Gaussian counterpart. The quantity

$$\varepsilon_r = \sup_{t \leqslant T} |D_r(t) - D(t)| + \sup_{s,t \leqslant T} |I_r(s, t) - I(s, t)|$$

controls the convergence, for (4.3) or (4.16) reads

$$d_A(W_r, W) \leqslant C(r-1+\varepsilon_r),$$

with a constant C depending only on T and, possibly, on H, worsening with H approaching 0 or 1. In vague terms, it means that, for $r \to 1$, uniform in $t \in [0, T]$:

1. a WM-process is *almost* Gaussian;

2. its covariance is *almost* equal to the covariance of the MBL-process;

and, thus, the entire process is *almost* a fBm. This is the factual meaning of the statement of [BL80].

REFERENCES

[AB85] M. Ausloss and D. H. Berman, A multivariate Weierstrass–Mandelbrot function, *Proc. R. Soc. Lond. A* **400**:331–350 (1985).

[BL80] M. V. Berry and Z. V. Lewis, On the Weierstrass–Mandelbrot fractal function, *Proc. R. Soc. Lond. A* **370**:459–484 (1980).

[Chu93] J. Chu, Conditional fractal simulation, sequential indicator simulation, and interactive variogram simulation, PhD Dissertation, 1993.

[GR80] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products [translation]* (Academic Press, New York, 1980).

[HB89] T. A. Hewett and R. A. Behrens, Conditional simulation of reservoir heterogeneity with fractals, *SPE Reservoir Research* (1989).

[Kaw72] T. Kawata, *Fourier Analysis in Probability Theory* (Academic Press, New York, 1972).

[Kle74] V. Klemeś, The Hurst phenomenon: A puzzle? *Water Resources Research* **10**(4):675–688 (1974).

[Kol40] A. Kolmogorov, Wienersche Spiralen und einige andere interessante Kurven im Hilbertischen Raum, *Doklady Acad. Sci. URSS, New Series* **26**:115–118, 1940.

[Kor92] G. Korvin, *Fractal Models in the Earth Sciences* (Elsevier, 1992).

[LK77] A. J. Lawrance and N. T. Kottegoda, Stochastic modelling of riverflow time series, *J. R. Statist. Soc. A* **140**:1–31 (1977).

[Lo91] A. W. Lo, Long term memory in stock market prices, *Econometrica* **59**(5):1279–1313 (1991).

[Lu70] Eugene Lukacs, *Characteristic Functions*, Second ed. (Hafner Publishing Co., New York, 1970).

[JS87] J. Jacod and A. Shiryaev, *Limit Theorems for Stochastic processes* (Springer, 1987).

[Man77] B. Mandelbrot, *Fractals: Form, Chance, and Dimension.* (Freeman and Co., San Francisco, 1977).

[MVN68] B. Mandelbrot and J. Van Ness, Fractional Brownian motion, fractional noises and applications, *SIAM Review* **10**:422–437 (1968).

[MW68] B. Mandelbrot and J. R. Wallis, Noah, Joseph, and operational hydrology, *Water Resources Research* **4**(5):909–918 (1968).

[MW69a] B. Mandelbrot and J. R. Wallis, Computer experiments with fractional Gaussian noises. Part 1, averages and variances, *Water Resources Research* **5**(1):228–267 (1969).

[MW69b] B. Mandelbrot and J. R. Wallis, Robustness of the rescaled range R/S analysis in the measurement of noncyclic long run statistical dependence, *Water Resources Research* **5**(5):967–988 (1969).

[MW69c] B. Mandelbrot and J. R. Wallis, Some long run properties of geophysical records, *Water Resources Research* **5**(2):321–340 (1969).

[MB93] F. J. Molz and G. K. Boman, A fractal based stochastic interpolation scheme in subsurface hydrology, *Water Resources Research* **29**(11):3769–3774 (1993).

[MSL97] F. J. Molz, J. Szulga, and H. H. Liu, Fractional Brownian motion and fractional Gaussian noise in subsurface hydrology: A review of fundamental properties, *Water Resources Research* **33**(10):2273–2286 (1997).

[PT00] V. Pipiras and M. S. Taqqu, Convergence of the Weierstrass–Mandelbrot process to fractional Brownian motion, *Fractals* **8**(4):369–384 (2000).

[Rac91] S. T. Rachev, *Probability Metrics and the Stability of Stochastic Models*, Series in Probability and Mathematical Statistics: Applied Probability and Statistics (Wiley, Chichester, 1991).

[Rü91] W. Rümelin, Simulation of fractional Brownian Motion, in *Fractals in the Fundamental and Applied Sciences*, Proceedings of the First IFIP Conference, Lisbon 1990, H.-O. Peitgen, J. M. Henriques and L. F. Penedo, ed. (North-Holland Publishing Co., Amsterdam, 1991).

[S98] J. Szulga, *Introduction to Random Chaos* (Chapman & Hall, 1998).

[SO1] J. Szulga, Hausdorff dimension of Weierstrass–Mandelbrot process, Auburn University Preprint, 2001.

[SM00] J. Szulga and F. J. Molz, On simulating fractional Brownian motion, in *High Dimensional Probability II*, E. Giné, et al., ed. (Birkhauser, Boston, 2000), pp. 377–387.

[Tur89] D. L. Turcotte, Fractals in geology and geophysics, *PAGEOPH* **131**(1/2):171–196 (1989).

[Vos89] R. F. Voss, Voss, Random fractals: self-affinity in noise, music, mountains, and clouds, *Fractals in Physics (Vence, 1989)* **38**(1-3):362–371 (1989).

[Yag58] A. M. Yaglom, Correlation theory of processes with random stationary increments, *Trans. Amer. Math. Soc.* **8**:87–141 (1958).